Intro GGT Homework 2

General suggestion: draw some graphs!

Question 1. Let G be a group that acts on a connected graph Γ.

1. Let $p = (v_0, \ldots, v_n)$ be a path in Γ. Show that for any $g \in G$, the sequence

$$g \cdot p := (gv_0, \ldots, gv_n)$$

   is also a path.

2. Let $g \in G \setminus \{1_G\}$ and let x be a vertex of Γ such that $d(x, gx)$ is minimal among all pairs (h, y) with $h \in G \setminus \{1_G\}$ and y a vertex of Γ. Let p be a geodesic (i.e. a shortest path) from x to gx.

   Show that for any $i \in \mathbb{Z}$ such that $g^i \neq 1_G$, the intersection of p and $g^i p$ can occur only at their endpoints. Moreover, if such an intersection occurs, then either $g^i = g$ or $g^i = g^{-1}$.

   (Hint: otherwise, let y be an intersection vertex and consider $g^i y$.)

3. Let $p = (v_0, \ldots, v_n)$ and $q = (w_0, \ldots, w_m)$ be two paths. The **concatenation** of p and q is defined when $v_n = w_0$ and is the path

$$p * q := (v_0, \ldots, v_n, w_1, \ldots, w_m).$$

   Let g and p be as in the previous item, and assume that g has finite order k and that p has length at least one. Show that the concatenation

$$p * gp * \cdots * g^{k-1}p$$

   forms either a simple cycle or a *backtrack cycle*, i.e. a path of the form (u, v, u). Determine precisely under which conditions the backtrack cycle occurs.

4. Conclude that if Γ is a tree, then any finite order element $g \in G$ always has either a fixed vertex or a fixed edge.

Question 2. Let F be a free group.

1. Find a tree on which F acts freely.

2. Justify your answer.

Question 3. We aim to prove that the converse of the previous question. That is, if a group G acts freely on a tree Γ, then G is a free group.

If Γ' is a subgraph of Γ we simply denote $\Gamma' \subset \Gamma$. We will need the following lemma.

Lemma (Zorn's lemma for subgraphs)**.** *Let $\mathcal{T}$ be a non-empty collection of subgraphs of a graph Γ. If every* chain *$\mathcal{C} \subset \mathcal{T}$ has an* upper bound *in $\mathcal{T}$, then $\mathcal{T}$ contains a* maximal *graph.*

Where:

- A subcollection $\mathcal{C} \subset \mathcal{T}$ is called a *chain* if for all $H_1, H_2 \in \mathcal{C}$, either $H_1 \subset H_2$ or $H_2 \subset H_1$.
- A graph $U \in \mathcal{T}$ is called an *upper bound* of a chain $\mathcal{C} \subset \mathcal{T}$ if $H \subset U$ for all $H \in \mathcal{C}$.
- A graph $M \in \mathcal{T}$ is called **maximal** in $\mathcal{T}$ if there is no graph $H \in \mathcal{T}$ such that $M \subset H$ and $M \neq H$.

In the following, let G be a group that **acts freely on a tree** Γ.

1. Let
$$\mathcal{T} := \{T \subset \Gamma \mid gT \cap T = \varnothing \; \forall g \in G \backslash \{1_G\}\}.$$
   (a) Show that $\mathcal{T}$ is non-empty. *(Hint: The action is free!)*
   (b) Let $\mathcal{C} \subset \mathcal{T}$ be a chain. Show that $\bigcup_{T \in \mathcal{C}} T \in \mathcal{T}$.
   (c) By Zorn's lemma, show that $\mathcal{T}$ contains a maximal graph.

2. Let M be a maximal graph in $\mathcal{T}$. We shall prove that $GM := \bigcup_{g \in G} gM$ has all the vertices of Γ.
   (a) Show that gM is also maximal in $\mathcal{T}$ for every $g \in G$.
   (b) Let u be a vertex of Γ. Show that $u \in GM$ if and only if $\mathrm{Orb}(u) \cap M \neq \varnothing$.
   (c) Suppose that there exists a vertex $v \notin GM$. Choose a vertex $u \in GM$ and let $p = (v_0, \ldots, v_n)$ be a path from $u = v_0$ to $v = v_n$.
   Show that there exists $0 \leq i \leq n-1$ such that $v_i \in GM$ and $v_{i+1} \notin GM$.
   (d) Let $g' \in G$ such that $v_i \in g'M$. Let M' be the graph obtained by adjoining the edge $\{v_i, v_{i+1}\}$ to $g'M$. Show that $M' \in \mathcal{T}$. Conclude a contradiction.

3. Let M be a maximal graph in $\mathcal{T}$. For distinct elements $g_1, g_2 \in G$, we say that g_1M and g_2M are **adjacent** if there exists and edge $e = \{u_1, u_2\}$ of Γ that connects them. That is, u_1 is a vertex of g_1M and u_2 is a vertex of g_2M. Let
$$\widetilde{S} = \{s \in G \mid M \text{ and } sM \text{ are adjacent}\}.$$
   (a) Show that if g_1M and g_2M are adjacent, there is only one edge that connects them.
   (Hint: A tree has no simple cycle.)
   (b) Show that $s \in \widetilde{S}$ if and only of $s^{-1} \in \widetilde{S}$.
   (c) According to Question 1, justify that G has no elements of order 2.
   Justify that we can choose $S \subset \widetilde{S}$ such that $S \cap S^{-1} = \varnothing$ and $S \cup S^{-1} = \widetilde{S}$.

4. Let $g \in G$ and $x \in M$. Consider the unique simple path from x to gx.
   (a) Why is the simple path unique?
   (b) Show that there exists a unique sequence $g_0, \ldots, g_n$ such that $g_0 = 1_G$, $g_n = g$, and g_iM is adjacent to $g_{i+1}M$ for all $0 \leq i \leq n-1$.
   (c) Deduce a map from $W(\widetilde{S})$, the set of words of $\widetilde{S} = S \cup S^{-1}$, to G.
   (d) Show that this map induces a well-defined isomorphism from $F(S)$ to G.